第五次作业报告

1、不定积分不要漏C!!! 好多人漏了,一定要注意!!!

2、设$f(x)=k\tan 2x$的一个原函数为$\frac{2}{3}\ln\cos 2x$ 求$k=$? $-\frac{4}{3}$

很多同学写了$-\frac{2}{3}$

解: $(\frac{2}{3}\ln\cos 2x)'=\frac{2}{3}\frac{1}{\cos 2x}\cdot(-\sin 2x)\cdot 2=-\overset{k}{\frac{4}{3}}\tan 2x$

复合函数求导法则 错了的一定课下巩固链锁法则.

3、$\int\frac{1}{\sqrt{1-25x^2}}dx$

方1: 凑微分 $d(5x)$

很多同学对三角换元不敏感

一定注意!!!

方2: $\sqrt{a^2-b^2x^2}$ 怎么去处理? 联想到 $1-\sin^2x=\cos^2x$

但是系数是a^2、b^2怎么办? ⇒ 换元 令$x=\frac{a}{b}\sin t$

$\sqrt{a^2-b^2x^2}=\sqrt{a^2-b^2\cdot\frac{a^2}{b^2}\sin^2t}=\sqrt{a^2(1-\sin^2t)}=a\cos t$

$\sqrt{a^2+b^2x^2}$ 怎么处理? 联想到 $1+\tan^2x=\sec^2x$

令$x=\frac{a}{b}\tan t$ $\sqrt{a^2+b^2x^2}=\sqrt{a^2+b^2\cdot\frac{a^2}{b^2}\tan^2t}=a\sec t$

这个也要会!!!

不要记公式,清理解

核心就是凑出 $1-\sin^2x$

$1+\tan^2x$

通过换元法 换系数

解: 令$x=\frac{1}{5}\sin t$ ⇒ $t=\arcsin 5x$

$I=\cdots\cdots$

4、$\int\frac{1}{x^2\sqrt{1+x^2}}dx$

方1: 看到$\sqrt{1+x^2}$ 按上思路 令$x=\tan t$ 可解

方2: 倒代换 (分母次数远高于分子)

令$x=\frac{1}{t}$

$I=\int\frac{-\frac{1}{t^2}dt}{\frac{1}{t^2}\sqrt{1+\frac{1}{t^2}}}=-\int\frac{t}{\sqrt{t^2+1}}dt=-\frac{1}{2}\int\frac{d(t^2+1)}{\sqrt{t^2+1}}=\cdots\cdots$

还有换元以后最后得代入回去!!! 别留t啥的, 最后要都是x

5、上次作业最后一题：

$$\int \sqrt{1-x^2}\arcsin x\,dx$$

三角换元要敏感

↓还是T3的思路 —— 三角换元!

令 $x=\sin t$

$$I=\int \cos t\cdot t\,d(\sin t)$$

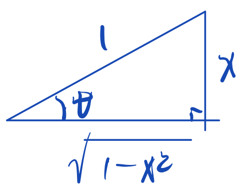
$$=\int \cos^2 t\cdot t\,dt$$

$$=\int \frac{1+\cos 2t}{2}\,t\,dt$$

$\cos 2t\cdot t$ 分部积分

$$=\frac{1}{4}t^2+\frac{1}{4}t\sin t+\frac{1}{8}\cos 2t+C$$

↓记住代入回去，画一个小小的Rt△



$\sin 2t=2\sin t\cos t=2x\sqrt{1-x^2}$

$\cos 2t=1-2\sin^2 t=1-2x^2$

$$=\frac{1}{4}(\arcsin x)^2+\frac{1}{4}\arcsin x\cdot 2\cdot x\cdot\sqrt{1-x^2}+\frac{1}{8}(1-2x^2)+C$$

$$=\frac{1}{4}(\arcsin x)^2+\frac{1}{2}x\sqrt{1-x^2}\arcsin x-\frac{1}{4}x^2+C$$

1.5.2 设 $f(x)=k\tan 2x$ 的一个原函数为 $\frac{2}{3}\ln\cos 2x$ 求 k.

解: $\left(\frac{2}{3}\ln\cos 2x\right)' = \frac{2}{3}\cdot\frac{1}{\cos 2x}\cdot(-\sin 2x)\cdot 2 = -\frac{4}{3}\tan 2x$

∴ $k=-\frac{4}{3}$

1.5.3 求 $f(x)$

(1) $\int f(x)\,dx = 3e^{\frac{x}{3}}+C$

解: $f(x) = \left(3e^{\frac{x}{3}}+C\right)' = e^{\frac{x}{3}}$

1.5.5 设 $\int f(x)\,dx = x^2+C$. 求 $\int x f(1-x^2)\,dx$

解: $f(x)=(x^2+C)'=2x$.

∴ $I=\int x\cdot 2\cdot(1-x^2)\,dx = \int 2x-2x^3\,dx = x^2-\frac{1}{2}x^4+C$

1.5.6 求下列不定积分

(3) $\int \frac{x^3-27}{x-3}\,dx$

用立方差公式 $a^3-b^3=(a-b)(a^2+ab+b^2)$ $\quad a^3+b^3=(a+b)(a^2-ab+b^2)$

$= \int \frac{\cancel{(x-3)}(x^2+3x+9)}{\cancel{x-3}}\,dx$

$= \int x^2+3x+9\,dx$

$= \frac{1}{3}x^3+\frac{3}{2}x^2+9x+C$

(5) $\int \frac{2x^2}{1+x^2}\,dx$

$= \int \frac{2(x^2+1)-2}{1+x^2}\,dx$

$= \int \left(2-\frac{2}{1+x^2}\right)dx$

$= 2x-2\arctan x+C$

(7) $\int \frac{1}{\sin^2 x\cos^2 x}\,dx$

$= \int \frac{\sin^2 x+\cos^2 x}{\sin^2 x\cos^2 x}\,dx$

$= \int \sec^2 x+\csc^2 x\,dx$

$= \tan x-\cot x+C$

1.5.7 用换元法计算

(1) $\int e^x\sqrt{3+2e^x}\,dx$

$=\frac{1}{2}\int\sqrt{3+2e^x}\,d(3+2e^x)$

$=\frac{1}{2}\cdot\frac{2}{3}(3+2e^x)^{\frac{3}{2}}+C$

$=\frac{1}{3}(3+2e^x)^{\frac{3}{2}}+C$

(3) $\int x\cot(x^2+1)\,dx$

$=\frac{1}{2}\int\cot(x^2+1)\,d(x^2+1)$

$=\frac{1}{2}\ln|\sin(x^2+1)|+C$

$\int\cot\,dx=\ln|\sin x|+C$

$\int\tan x\,dx=-\ln|\cos x|+C$

(8) $\int\frac{1}{x^2+2x+2}\,dx$

$=\int\frac{1}{(x+1)^2+1}\,d(x+1)$

$=\arctan(x+1)+C$

(10) $\int\frac{1}{\sqrt{1-25x^2}}\,dx$

方1: $=\frac{1}{5}\int\frac{1}{\sqrt{1-(5x)^2}}\,d(5x)$

$=\frac{1}{5}\arcsin 5x+C$

方2: 令 $x=\frac{1}{5}\sin t \Rightarrow t=\arcsin 5x$

$I=\int\frac{1}{\cos t}\,d(\frac{1}{5}\sin t)=\frac{1}{5}\int 1\,dt=\frac{1}{5}t+C=\frac{1}{5}\arcsin 5x+C$

(17) $\int\frac{1}{x\sqrt{a^2-x^2}}\,dx$

$\int\frac{1}{\cos x}\,dx=\int\sec x\,dx=\ln|\sec x+\tan x|+C$

$\int\frac{1}{\sin x}\,dx=\int\csc x\,dx=\ln|\csc x-\cot x|+C$

令 $x=a\sin t \Rightarrow t=\arcsin\frac{x}{a}$

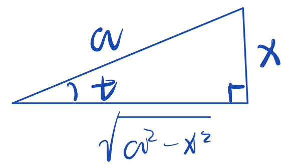
$I=\int\frac{1}{a\sin t\cdot a\cos t}\,d(a\sin t)$

$=\int\frac{\cancel{a}\cdot\cancel{\cos t}}{a\sin t\cdot\cancel{a}\cdot\cancel{\cos t}}\,dt$

$=\frac{1}{a}\int\frac{1}{\sin t}\,dt$

很重要：画小三角法

$x = a\sin t \Rightarrow \sin t = \frac{x}{a}$



$\cot t = \frac{\sqrt{a^2-x^2}}{x}$

$\csc x = \frac{a}{x}$

$$= \frac{1}{a}\ln|\csc x - \cot x| + C$$

$$= \frac{1}{a}\ln\left|\frac{a-\sqrt{a^2-x^2}}{x}\right| + C$$

(17) $\int \frac{1}{x^2\sqrt{1+x^2}}\,dx$

方1：倒代换 令 $x=\frac{1}{t}$ 分母次数远高于分子时，且尽量不出现常数

解释一下：$\frac{1}{(x+1)\sqrt{x^2+3}}$

不太好用

$$I = \int \frac{1}{\frac{1}{t^2}\sqrt{1+\frac{1}{t^2}}}\,d\left(\frac{1}{t}\right)$$

$$= \int \frac{-\frac{1}{t^2}}{\frac{1}{t^2}\sqrt{1+\frac{1}{t^2}}}\,dt$$

$$= -\int \frac{t}{\sqrt{t^2+1}}\,dt$$

$$= -\frac{1}{2}\int \frac{d(t^2+1)}{\sqrt{t^2+1}}$$

$$= -\frac{1}{2}\cdot 2\cdot (t^2+1)^{\frac{1}{2}} + C$$

$$= -\sqrt{\frac{1}{x^2}+1} + C$$

$$= -\sqrt{\frac{1+x^2}{x^2}} + C$$

方2：令 $x=\tan t$

$$I = \int \frac{1}{\tan^2 t\sqrt{1+\tan^2 t}}\,d\tan t$$

$$= \int \frac{\sec^2 t\,dt}{\tan^2 t\sec t} = \int \frac{\cos t\,dt}{\sin^2 t} = \int \frac{d(\sin t)}{\sin^2 t} = -\frac{1}{\sin t} + C$$

$$= -\frac{\sqrt{1+x^2}}{x} + C$$

(18) $\int \frac{1}{(x+2)\sqrt{1+x}}dx$

令 $\sqrt{x+1}=t \quad x=t^2-1$

$I=\int \frac{d(t^2-1)}{(t^2+1)\cdot t}$

$=\int \frac{2t\,dt}{t(t^2+1)}$

$=2\arctan t+C$

$=2\arctan\sqrt{1+x}+C$

1.5.9 用分部积分计算

(1). $\int x^2e^{-x}dx$

$=-\int x^2 de^{-x}$

$=-[x^2e^{-x}-\int e^{-x}\cdot(2x)dx]$

$=-[x^2e^{-x}+\int 2x\,d(e^{-x})]$

$=-[x^2e^{-x}+2xe^{-x}-\int e^{-x}d(2x)]$

$=-[x^2e^{-x}+2xe^{-x}+2e^{-x}+C]$

$=-x^2e^{-x}-2xe^{-x}-2e^{-x}+C$

(5). $\int x^2\cos x\,dx$

$=\int x^2 d\sin x$

$=x^2\sin x-2\int \sin x\cdot x\,dx$

$=x^2\sin x+2\int x\,d\cos x$

$=x^2\sin x+2(x\cos x-\int \cos x\,dx)$

$=x^2\sin x+2x\cos x-2\sin x+C$

(8). $\int \ln\frac{x}{2}dx$

$=\ln\frac{x}{2}\cdot x-\int x\cdot d(\ln\frac{x}{2})$

$=\ln\frac{x}{2}\cdot x-\int x\cdot\frac{1}{\frac{1}{2}x}\cdot\frac{1}{2}dx$

$=\ln\frac{x}{2}\cdot x-x+C$

(10). $\int \arcsin x \, dx$

$= \arcsin x \cdot x - \int x \cdot \frac{1}{\sqrt{1-x^2}} \, dx$

$= \arcsin x \cdot x + \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}} \, d(1-x^2)$

$= \arcsin x \cdot x + \frac{1}{2} \cdot 2 \cdot \sqrt{1-x^2} + C$

$= \arcsin x \cdot x + \sqrt{1-x^2} + C$

1.5.10

(1). $\int \arctan\sqrt{x} \, dx$

$= x\arctan\sqrt{x} - \int x \, d(\arctan\sqrt{x})$

$= x\arctan\sqrt{x} - \frac{1}{2}\int \frac{\sqrt{x}}{1+x} \, dx$

单独研究 $\int \frac{\sqrt{x}}{1+x} dx$

令$\sqrt{x}=t$ ∴ $x=t^2$ $I = \int \frac{t \cdot 2t}{1+t^2} dt = \int \frac{2(t^2+1)-2}{t^2+1} dt = 2t - 2\arctan t + C = 2\sqrt{x} - 2\arctan\sqrt{x} + C$

∴原积分 $= x\arctan\sqrt{x} - \frac{1}{2} I$

$= x\arctan\sqrt{x} - \sqrt{x} + \arctan\sqrt{x} + C$

(3) $\int \sqrt{1-x^2} \arcsin x \, dx$

令 $x = \sin t$

$= \int \cos t \cdot t \, d(\sin t)$

$= \int \cos^2 t \cdot t \, dt$

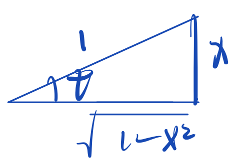
$= \int \frac{1+\cos 2t}{2} \cdot t \, dt$

$= \frac{1}{4}t^2 + \frac{1}{2}\int \cos 2t \cdot t\,dt$ 分部积分

$= \frac{1}{4}t^2 + \frac{1}{4}\int t \, d\sin 2t$

$= \frac{1}{4}t^2 + \frac{1}{4}(t\sin 2t - \int \sin 2t \, dt)$

$= \frac{1}{4}t^2 + \frac{1}{4}t\sin 2t + \frac{1}{8}\cos 2t + C$



$\sin 2t = 2\sin t \cos t = 2 \cdot x \cdot \sqrt{1-x^2}$

$\cos 2t = 1 - 2\sin^2 t = 1 - 2x^2$

$= \frac{1}{4}(\arcsin x)^2 + \frac{1}{4}\arcsin x \cdot 2 \cdot x \cdot \sqrt{1-x^2} + \frac{1}{8}(1-2x^2) + C$ $\quad \frac{1}{8}+C \to C$

$= \frac{1}{4}(\arcsin x)^2 + \frac{1}{2}x\sqrt{1-x^2}\arcsin x - \frac{1}{4}x^2 + C$